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Graphs and topologies on discrete sets

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Abstract

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We show that a graph admits a topology on its node set which is compatible with the usual connectivity of undirected graphs if, and only if, it is a comparability graph. Then, we give a similar condition for the weak connectivity of oriented graphs and show there is no topology which is compatible with the strong connectivity of oriented graphs. We also give a necessary and sufficient condition for a topology on a discrete set to be 'representable' by an undirected graph.

Résumé

Nous montrons qu'un graphe admet une topologie sur l'ensemble de ses sommets compatible avec la connexité usuelle des graphes non-orientés si, et seulement si c'est un graphe de comparabilité; puis nous donnons une condition similaire pour la connexité faible des graphes orientés et montrons la non-existence d'une topologie compatible avec la connexité forte. Nous donnons également une condition nécessaire et suffisante pour qu'une topologie sur un ensemble discret soit 'représentable' par un graphe non-orienté.

1. Introduction

Given a graph, our aim is to define a topology (we will call *compatible*) on this graph such that an induced subgraph is connected (in the usual meaning) if, and only if, it is connected for this topology. This question has not been studied yet in general (see [3] for a particular case).

It happens often that the object represented by a graph is in fact represented by the nodes of this graph; the edges then are just indicating the neighbouring relationship between the nodes (for example to represent the plane or the space by a graph [3, 7, 8]). Thus, it seems interesting to define a compatible topology on the set of the nodes only. In addition, this respects the discrete nature of graphs.

In this paper, we shall consider finite and infinite graphs, but with a finite number of connected components and such that each node has a finite degree. The topologies will all be defined on node sets. If $G=(N, A)$ is defined as a graph; N is the node set, and A the edge set if G is undirected and the arc set if G is oriented. The nodes will be noted $x, y, \ldots,$ the edges $\{x, y\}$, the arcs (x, y).

2. The undirected graphs

The main result of this section is the equivalence between graphs which admit a compatible topology and comparability graphs.

The comparability graph of an ordered set $(X, <)$ is the graph $G=(X, A)$, with $\{x, y\}$ in A whenever $x<y$ or $y<x$. We shall rather use the following equivalent definition: a comparability graph is a simple undirected graph G which can be oriented in such a way that if there exists a path from x to y, then (x, y) is an arc. We shall call such an orientation *compatible orientation* of G. Comparability graphs have a great importance in graph theory and combinatorics. They have been studied for theoretical problems [5] or algorithmic aspects [4, 6].

First, we shall characterise the compatible topologies of a graph. The we will give conditions, concerning a topology on a discrete set, for the existence of an undirected graph admitting this topology as a compatible one.

Property 1. *Let T be a topology on the node set of a graph. The topology T is compatible if, and only if*

(a) *$\{x, y\}$ is an edge if, and only if*

$$\text{every open set containing } x \text{ contains } y \quad \text{or}$$
$$\text{every open set containing } y \text{ contains } x, \tag{1}$$

(b) *for every node x, there exists an open set $o(x)$ containing x and no node which is not a neighbour of x.*

Proof. Let us suppose that the topology T is a compatible one. Let $\{x, y\}$ be an edge of G. If there exists an open set O containing x and not y, and an open set O' containing y and not x, the restriction of O and O' to $\{x, y\}$ would be a partition of $\{x, y\}$ in two disjunctive open sets. This is impossible, since the set of the two endvertices of an edge is connected. If x and y are two nodes which are not neighbours, the set $\{x, y\}$ does not induce a connected subgraph. Thus (1) is not verified and (a) is true.

Let x be a node and X the set of all the nodes, different from x, which are not neighbours of x. The connected components of X are, on the one hand, the connected components of G (except for that of x), and on the other hand the components obtained from the component of x (in G). But these components are at most

$$\sum_{y \in V(x)} (\deg(y) - 1),$$

where $V(x)$ is the set of neighbours of x. Thus X has a finite number of connected components $X_1, \ldots, X_n$. For every $i \leq n$, $\{x\} \cup X_i$ is not connected, so there exists a partition of it in two open sets O and O'. If neither O nor O' is equal to $\{x\}$, their restriction to X_i is a partition of X_i into two disjoint open sets. So there exists an open set containing x and no nodes of X_i. We can take the intersection of all these open sets for $o(x)$.

Thus, the condition is necessary; let us show that it is sufficient.

Let T be a topology such that $\{x, y\}$ is an edge if, and only if, (1) is verified and for every node x, the open set $o(x)$ exists.

Let X be a set of nodes inducing a connected (in the usual meaning) subgraph. Suppose there exist $X_1 = O_1 \cap X$ and $X_2 = O_2 \cap X$ with O_1 and O_2 in T such that $X_1 \cap X_2 = \emptyset$. There exists $\{x, y\}$ in $X_1 \times X_2$ such that x and y are neighbours. Since (1) is verified, this partition of A is impossible.

Conversely, let X be a set of nodes inducing a nonconnected subgraph. $X = X_1 \cup X_2$ and no node of X_1 is a neighbour of a node of X_2. Let

$$O_1 = \bigcup_{x \in X_1} o(x) \quad \text{and} \quad O_2 = \bigcup_{x \in X_2} o(x).$$

O_1 and O_2 are open sets such that $(O_1 \cap X) \cap (O_2 \cap X) = \emptyset$ and $O_1 \cup O_2 \supset X$. Thus the topology T is compatible and the condition is sufficient. □

We remark that a compatible topology is not separable. On the other hand, part (a) of the condition can also be written as: *the edges are the only connected induced subgraphs containing exactly two nodes.*

Property 2. *Let T be a compatible topology and $\{x, y\}$ an edge of G. If every open set containing x contains y and every open set containing y contains x, then x and y have the same neighbours.*

Proof. Let z be a neighbour of y. (1) is true for (y, z).

If every open set containing y contains z, every open set containing x contains z. Thus, it is impossible to partition (x, z) into two disjoint open sets. Therefore, (x, z) is connected and z is a neighbour of x.

If every open set containing z contains y, every open set containing z contains x, and z is a neighbour of x.

Likewise, if z is a neighbour of x, $\{y, z\}$ is an edge. Thus, x and y have the same neighbours. □

Theorem. *Let T be a topology on a discrete set N. There exists an undirected graph $G = (N, A)$, with a finite number of connected components, such that every* *node has finite degree and admitting T as a compatible topology if, and only if*

$$\forall x \in N, \exists O_x \in T, O_x \text{ finite}, \forall O \in T, (x \in O) \Rightarrow (O \supset O_x).$$

Proof. Let us prove that the condition is necessary. By Property 1, if a topology is compatible with the connectivity of a graph, for every node x, there exists an open set $o(x)$ containing only x and neighbours of x. The set $o(x)$ is finite (like degree (x)). Let us suppose that there exists an open set O containing x and a node y which is in $o(x)$ but not in O. The node y is a neighbour of x. Let $o'(x)$ be the open set $O \cap o(x)$. We can build the open set O_x by repeating this operation at most degree (x) times.

Thus the condition is necessary; let us show it is sufficient.

Let us suppose the condition is verified. We build the graph G by

$$\{x, y\} \in A \Leftrightarrow ((y \in O_x) \quad \text{or} \quad (x \in O_y)).$$

Let X be a subset of N inducing a connected subgraph. Let us suppose there exists a partition of X into two disjoint open sets O and O'. For every node x of O, O_x is included in O. The open set O' has the same property. Thus

$$O = \bigcup_{x \in O} O_x, \qquad O' = \bigcup_{x \in O'} O_x.$$

There exists an edge $\{x, y\}$ with x in O and y in O'. Thus x is in O_x or y in O_x. In both cases, $O \cap O'$ is different from $\emptyset$.

Let X be a set of nodes inducing a nonconnected subgraph. There exists a partition of X in $X_1 \cup X_2$ such that no node of X_1 is neighbour with a node of X_2. We can define two open sets by

$$O = \bigcup_{x \in X_1} O_x, \qquad O' = \bigcup_{x \in X_2} O_x.$$

$O \cap X_1$ contains no node of X_2, and $O' \cap X_2$ contains no node of X_1. Thus we have a partition of X into two disjoint open sets. So, the condition is sufficient. □

Remark. If the graph G is not locally finite or has an infinite number of connected components, the condition is always sufficient (the proof remains true), but it is no longer necessary. Let us consider for example the graph $G = (N, A)$

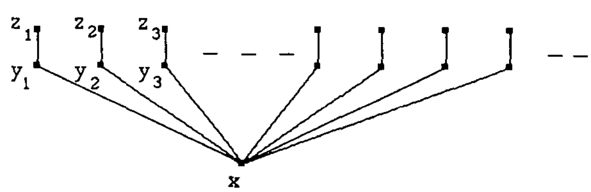

Fig. 1.

defined by (see Fig. 1)

$$N = \{x\} \cup \{y_n, n \in \mathbb{Z}^{*+}\} \cup \{z_n, n \in \mathbb{Z}^{*+}\},$$
$$A = \{\{x, y_n\}, n \in \mathbb{Z}^{*+}\} \cup \{\{y_n, z_n\}, n \in \mathbb{Z}^{*+}\}.$$

Let T be the topology generated by $T_1 \cup T_2 \cup T_3 \cup \{\emptyset\}$, with

$$T_1 = \{\{y_n\}, n \in \mathbb{Z}^{*+}\},$$
$$T_2 = \{\{x\} \cup \{y_n, n \in \mathbb{Z}^{*+}\} \cup \{z_{k \cdot n}, n \in \mathbb{Z}^{*+}\}, k \in \mathbb{Z}^{*+}\},$$
$$T_3 = \{\{y_n, z_n\}, n \in \mathbb{Z}^{*+}\}.$$

Let us show that T is a compatible topology. First, we remark that the intersection of an open set O of T_1 with another open set is O or the empty set. The intersection of an open set O of T_3 with an open set of T_2 or T_3 is O, the empty set or an open set of T_1. The intersection of two open sets of T_2 is an open set of T_2:

$$(\{x\} \cup \{y_n, n \in \mathbb{Z}^{*+}\} \cup \{z_{k \cdot n}, n \in \mathbb{Z}^{*+}\}) \cap (\{x\} \cup \{y_n, n \in \mathbb{Z}^{*+}\} \cup \{z_{p \cdot n}, n \in \mathbb{Z}^{*+}\})$$
$$= \{x\} \cup \{y_n, n \in \mathbb{Z}^{*+}\} \cup \{z_{q \cdot n}, n \in \mathbb{Z}^{*+}\}$$

where $q = \text{gcm}(p, k)$. Thus, there exists no minimal open set containing x.

Let X be a subset of N inducing a nonconnected subgraph. If X does not contain x, then X is included in a (disjoint) union of open sets of T_3. If x is in X, there exists k in $\mathbb{Z}^{*+}$ such that z_k is in X but not y_k. Let us consider the two following open sets:

$$O = \{y_k, z_k\},$$
$$O' = \bigcup_{p<k} \{y_p, z_p\} \cup \bigcup_{p>k} (\{x\} \cup \{y_n, n \in \mathbb{Z}^{*+}\} \cup \{z_{p \cdot n}, n \in \mathbb{Z}^{*+}\}).$$

We have, $O \cup O' \supset X$, and $(X \cap O) \cap (X \cap O') = \emptyset$.

Let X be a subset of N inducing a connected subgraph. If x is not in X, then X is made of only one node or is equal to $\{y_k, z_k\}$ for one k in $\mathbb{Z}^{*+}$. And all open sets containing z_k contain y_k. Otherwise, for every k in $\mathbb{Z}^{*+}$, if z_k is in X, y_k is also in X. Let us suppose there exists a partition of X into two open sets O and O', with x in O. If y_k is in X, y_k is in O (the open sets of T_2 contain all the nodes y_k). In addition, if z_k is in O', y_k is in O'. Thus this partition of X is impossible.

Theorem. *The graph $G = (N, A)$ admits a compatible topology on N if, and only if, G is a comparability graph.*

Proof. Let T be a compatible topology. We orient A as follows:

• if $\{x, y\}$ is an edge such that every open set containing x contains y and not every open set containing y contains x, then (x, y) is an arc.

• if $X = \{x_i, i \in I\}$ is a set of nodes such that every open set containing x_i contains x_j, (in this case, by Property 2, all these nodes are neighbours and have the same neighbours; we take X maximal for this property; we remark that X is an equivalence class for this relation). Since all nodes are of finite degree, I is a finite set $[1, \ldots, n]$. The pair (x_i, x_j) is an arc if, and only if, $j > i$.

Suppose there exists a path $((x, y), (y, z))$. Every open set containing x contains z. If not every open set containing z contains x, then (x, z) is an arc. If every open set containing z contains x, then every open set containing y contains x; the three nodes are in the same equivalence class. In addition, in the indexation of this class, the index of x is smaller than the index of y, which is smaller than that of z. Thus (x, z) is an arc. By induction, if there exists a finite oriented path from x to z, then (x, z) is an arc. Since every node is of finite degree, we do not need to consider infinite paths. So, G is a comparability graph and the condition is necessary.

Conversely, let $G = (N, A)$ be a comparability graph, and $G' = (N, A')$ be the graph obtained by a compatible orientation of G.

For every node x, let us define

$$O_x = \bigcup_{y/(x,y)\in A} \{x, y\} \quad \text{and} \quad O = \{O_x\}_{x\in N}.$$

Let T be the set of unions of sets of O. Let us show that T is a compatible topology.

The union of all the elements of T is N.

The intersection of two elements of O is in T: Let O_x and O_y be in O. If x and y are neighbours, we may suppose that $y \in O_x$. Let z be in O_x. (x, y, z) is a path, so z is in O_x. In this case, $O_x \cap O_y = O_y$. If x and y are not neighbours, let z be in $O_x \cap O_y$ and i in O_z. (x, z, i) and (y, z, i) are paths, so i is in $O_x \cap O_y$. In this case, $O_x \cap O_y$ is a union of sets of O. Thus T is a topology.

The topology T is compatible: For every node x, all the open sets containing x contain O_x, which is an open set containing only x and neighbours of x. In addition, if x and y are neighbours, we have x in O_y or y in O_x. $\square$

Remark. Let $(X, <)$ be a discrete ordered set. The order topology of X is not compatible with the comparability graph of $(X, <)$. Let us suppose that, for every x in X, there exists y in X such that $x < y$ or $y < x$. The smallest open set containing x is $\{x\}$. Thus, the order topology is compatible with the graph $G = (X, \emptyset)$.

If there exists x in X such that there is no node y with $x < y$ or $y < x$, the only open set containing x is X. Let Y be the set of the points of X which verify this property. In this case, the order topology is compatible with the graph $G = (X, A)$, where $A = \{\{x, y\}, x \in Y\}$.

3. The oriented graphs

For the oriented graphs, we can define two connectivities: the weak one (for every x and y in a connected component, there exists a (finite) path from x to y *or*

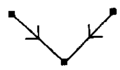

Fig. 2.

from y to x) and the strong one (there exists a finite path from x to y *and* from y to x). First, we will study the case of the weak connectivity.

A necessary condition for a graph to admit a compatible topology is that connected components are disjoint; in the case of weak connectivity, this is not always verified, for example for the graph of Fig. 2.

In order to define a compatible topology on a graph $G = (N, A)$, it is necessary that, for every induced subgraph $G' = (N', A')$, and every (x, y, z) in $N'3$ if there exists a path C in G' between x and y and a path C' in G' between x and z, then there exists a path (in G') between y and z. We can, without any loss of generality, suppose that N' is the set of the nodes of C and C'.

Let $H_G = (N, E)$ be the undirected graph obtained from G by

$$\{x, y\} \in E \text{ if and only if, } (x, y) \in A \quad \text{or} \quad (y, x) \in A.$$

Property 3. *Let $G = (N, A)$ be an oriented graph; if the weak connected components of G are disjoint, the connected components of H_G are exactly the weak connected components of G.*

Proof. Let x and y in N, and $(u_0 = x, \ldots, u_n = y)$ a path of H_G; let us show by induction on n that there exists a path of $G(u_{\alpha_0}, \ldots, u_{\alpha_n})$ with $(u_{\alpha_0}, u_{\alpha_n}) = (x, y)$. This is obvious for $n = 1$.

Let us suppose that it it true for $p < n$ $(n > 1)$, and let $U = (u_0 = x, \ldots, u_{n-1}, u_n = y)$ be a path of H_G; there exists paths in G joining x and u_{n-1}, and joining y and x_{n-1} containing only nodes of U. Since the weak connected components of G are disjoint, there exists an oriented path joining x and y. □

Thus we have the following.

Theorem. *Let $G = (N, A)$ be an oriented graph. The graph G admits a topology which is compatible with the weak connectivity if, and only if*

(i) *the weak connected components of G are disjoint,*

(ii) *H_G is a comparability graph.*

For the strong connectivity of an oriented graph $G = (N, A)$, there exist two trivial cases where a compatible topology can be defined. On the one hand, the case in which all the strongly connected components of G are made of one node. On the other hand, the case

$$((x, y) \in A) \Leftrightarrow ((y, x) \in A))$$

which is the case of undirected graphs.

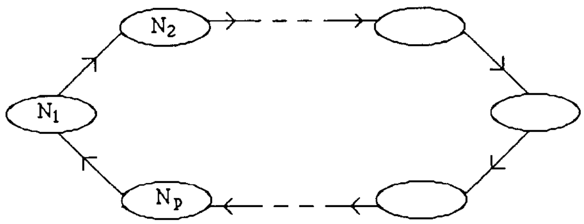

Fig. 3.

In order to avoid these two cases, we define a *true strongly connected component* of G as an induced subgraph $G' = (N', A')$ with $N' = N_1 \cup \cdots \cup N_p$, $p \geq 3$, such that (see Fig. 3) the subgraphs induced by $N_1, \ldots, N_p$ are strongly connected, and

$$(\exists (x, y) \in A \cap (N_i \times N_j)) \Leftrightarrow (j = i + 1 \pmod p))$$

Theorem. *Let G be an oriented graph having a true strongly connected component. Then no topology on N is compatible with strong connectivity.*

Proof. Let $G' = (N', A')$, with $N' = N_1 \cup N_2 \cup \cdots \cup N_p$, be a true strongly connected component of G. Each N_i induces a connected subgraph, like $\bigcup_{1 \leq i \leq p} N_i$. On the other hand, for every $i < p$, the subgraph induced by $N_i \cap N_p$ is not strongly connected. Thus, there exist, for every $i < p$, two open sets O_i and O_i' such that

$$(O_i \cup O_i') \supset (N_p \cup N_i), \qquad (N_p \cup N_i) \cap O_i \cap O_i' = \emptyset.$$

Since N_i is connected, if $N_i \cap O_i \neq \emptyset$, then $O_i \supset N_i$ (otherwise, we would have a partition of N_i into two disjoint open sets). So, we may assume that, for every $i < p$, $O_i \supset N_i$, $O_i' \supset N_p$, $N_i \cap O_i' = \emptyset$, and $N_p \cap O_i = \emptyset$. $\bigcap_{i<p} O_i'$ is an open set and we have

$$\bigcap_{i<p} O_i' = N_p.$$

$\bigcup_{i<p} O_i$ is an open set and we have

$$\bigcup_{i<p} O_i = \bigcup_{i<p} N_i.$$

We have a partition of N' into two disjoint open sets, which is impossible. So, strong connectivity admits no compatible topology. □

We remark that this last result remains true even for a topology which is not defined on N.

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